

# G-THEORY OF ROOT STACKS AND EQUIVARIANT K-THEORY

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ABSTRACT. Using the description of the category of quasi-coherent sheaves on a root stack in [BV], we study the  $G$ -theory of root stacks via localisation methods. We apply our results to the study of equivariant  $K$ -theory of algebraic varieties under certain conditions. In the process we prove a generalisation of the main result of [EL].

## 1. INTRODUCTION

Let  $X$  be an algebraic variety equipped with an action of a finite group  $G$ . In algebraic geometry one frequently needs to consider equivariant objects on  $X$  with respect to the action of  $G$ . These objects correspond to objects over the quotient stack  $[X/G]$ . However, it can happen that  $[X/H] \cong [X'/H']$  for seemingly unrelated  $X$  and  $X'$ . In such situation, it is useful to have a canonical description of the quotient stack  $[X/H]$ , perhaps in terms of its coarse moduli space  $Y$ . This may not always be possible but sometimes it is. In this paper we will describe a situation in which this occurs, see (4.11). When our hypothesis are satisfied the quotient stack becomes a root stack over its coarse moduli space  $Y$ .

Using the description of quasi-coherent sheaves on a root stack, see [BV], we give a description of the algebraic  $G$ -theory of a root stack. The main tool is the localisation sequence associated to a quotient category.

These results have immediate applications to equivariant  $K$ -theory. We obtain a generalization of the main result of ([EL]). This paper studies the equivariant Grothendieck group of a smooth curve. Combining (3.30) and (4.11) yields a generalisation of this theorem by noting that the hypothesis of (4.7) will always be satisfied for tame actions of groups on smooth projective curves. It should be noted that other approaches to equivariant  $K$ -theory using a top down description of the stack  $[X/G]$  exist in the literature, see [Vi].

After a short preliminary section §2, we start by studying the  $G$ -theory of a root stack in §3. The main tool is a description of the

category of quasi-coherent sheaves on a root stack, see [BV]. After recalling this description we apply localisation methods to compute the  $G$ -theory of a root stack over a noetherian scheme. The results are given in (3.27) and (3.4).

In §4 we address the issue of when a quotient stack is a root stack. First we show that under our assumptions (tameness of the action and ramification divisor is normal crossing) inertia groups of all points will be abelian (see Theorem 4.2). We use Luna's étale slice theorem (see [D]) and Chevalley-Sheppard-Todd theorem ([Bou]) in the proof. Then under the same hypothesis, we show that a quotient stack is a root stack, see (4.11). The main tool here is a generalisation of Abhyankar's lemma, see [SGA1, XIII, appendix I].

The paper ends in §5 by combing the results of the previous two sections to study equivariant  $K$ -theory of a scheme.

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## NOTATIONS AND CONVENTIONS

$k$	our base field
$\ker$	the kernel of a functor, see (2.2)
$\vec{r}$	an $n$ -tuple $(r_1, \dots, r_n)$ of real numbers
$[\vec{r}, \vec{s}]$	the poset of integer points in $\prod_{i=1}^n [r_i, s_i]$
$\vec{r}I^n$	the poset of integer points in $\prod_{i=1}^n [0, r_i]$
$\text{Func}(\mathbf{A}, \mathbf{B})$	functor category between two abelian categories
$\widehat{M}$	the dual $\text{Hom}(M, \mathbb{G}_m)$ of the monoid $M$
$X_{L, \vec{r}}$	a stack of roots over the scheme $X$
$\mathfrak{Coh} X$	category of coherent sheaves on $X$
$\mathcal{EP}(X, L, \vec{r})$	category of coherent extendable pairs, see (3.16)

## 2. LOCALIZATION VIA SERRE SUBCATEGORIES

**2.1. Serre subcategories.** Let  $\mathbf{A}$  be an abelian category. Recall that a *Serre subcategory*  $S$  of  $\mathbf{C}$  is a non-empty full subcategory that is closed under extensions, subobjects and quotients. When  $\mathbf{A}$  is well-powered the quotient category  $\mathbf{A}/S$  exists, see [S, pg. 44, Theorem 2.1].

We will need the following result to identify quotient categories.

**Theorem 2.1.** *Let  $F : \mathbf{A} \rightarrow \mathbf{B}$  be an exact functor between abelian categories. Denote by  $S$  the full subcategory whose objects are  $x$  with  $F(x) \cong 0$ . Then  $S$  is a Serre subcategory and we have a factorisation*

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbf{A}/\mathbf{C} \\ \downarrow F & \nearrow & \\ \mathbf{B} & & \end{array}$$

*Proof.* See [S, page 114] □

**Definition 2.2.** The category  $S$  is called the *kernel of the functor  $F$*  and is denoted by  $\ker(F)$ .

**Theorem 2.3.** *In the situation of the previous theorem suppose that we have*

- (1) *for every object  $y \in \mathbf{B}$  there is a  $x \in \mathbf{A}$  such that  $F(x)$  is isomorphic to  $y$  and*
- (2) *for every morphism  $f : F(x) \rightarrow F(x')$  there is  $x'' \in \mathbf{A}$  with  $h : x'' \rightarrow x$  and  $g : x'' \rightarrow x'$  such that  $F(h)$  is an isomorphism and the following diagram commutes*

$$\begin{array}{ccc} F(x'') & & \\ \downarrow F(h) & \searrow F(g) & \\ F(x) & \xrightarrow{f} & F(x'). \end{array}$$

*Then there is an equivalence of categories  $\mathbf{A}/\mathbf{C} \cong \mathbf{B}$ .*

*Proof.* See [S, pg. 114, theorem 5.11]. □

**2.2. Some functor categories.** Consider  $n$ -tuples of integers  $\vec{r} = (r_1, r_2, \dots, r_n)$  and  $\vec{s} = (s_1, s_2, \dots, s_n)$ . We denote by  $[\vec{r}, \vec{s}]$  the poset of  $n$ -tuples  $(x_1, \dots, x_n)$  with

$$x_i \in \mathbb{Z} \quad \text{and} \quad r_i \leq x_i \leq s_i.$$

We will make use of the following shorthand notation :

$$rI = [0, r] \quad \text{and} \quad \vec{r}I^n = [0, \vec{r}].$$

These intervals are naturally posets with

$$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n) \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for all } i.$$

This poset structure allows us to view them as categories in the usual way.

Fix an abelian category  $\mathbf{A}$  and consider the functor category

$$\text{Func}(\vec{r}I^n, \mathbf{A}).$$

This category is an abelian category with kernels, cokernels formed pointwise. We will be interested in the  $K$ -theory of such categories. In this subsection we will try to understand some of their quotient categories. Given an object  $\mathcal{F}$  in this category and an object  $u$  of  $\vec{r}I^n$  we denote by  $\mathcal{F}_u \in \mathbf{A}$  the value of the functor  $\mathcal{F}$  on this object and if  $u \leq v$  the arrow from  $F_u$  to  $F_v$  will be denoted by

$$F_{+(v-u)} : F_u \rightarrow F_v.$$

In particular, we take  $e_i = (0, 0, \dots, 1, 0, \dots, 0)$  to be a standard basis vector so that we have a morphism

$$F_{+e_i} : F_{(u_1, \dots, u_n)} \rightarrow F_{u_1, \dots, u_{i-1}, u_i+1, u_{i+1}, \dots, u_n}.$$

**Lemma 2.4.** *To give an object  $F$  of  $\text{Func}(\vec{r}I^n, \mathbf{A})$  is the same as providing the following data :*

- (D1) objects  $F_{(u_1, u_2, \dots, u_n)} \in \mathbf{A}$
- (D2) arrows

$$F_{+e_i} : F_u \rightarrow F_{u+e_i},$$

such that all diagrams of the form

$$\begin{array}{ccc} F_u & \xrightarrow{\quad\quad\quad} & F_{u+e_j} \\ \downarrow & & \downarrow \\ F_{u+e_i} & \xrightarrow{\quad\quad\quad} & F_{u+e_i+e_j} \end{array}$$

*Proof.* The hypothesis insure that if  $u \leq v$  in  $\vec{r}I^n$  then there is a well defined map  $F_u \rightarrow F_v$  which produces our functor.  $\square$

**Proposition 2.5.** (i) *Let  $\text{tr}_{n-1}(\vec{r}) = (r_1, r_2, \dots, r_{n-1})$ . There is an exact functor*

$$\pi : \text{Func}(\vec{r}I^n, \mathbf{A}) \rightarrow \text{Func}(\text{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$$

*defined on objects by*

$$\pi(G)_{(u_1, u_2, \dots, u_{n-1})} = (G)_{(u_1, \dots, u_{n-1}, 0)}$$

- (ii) *The functor  $\pi$  has a left adjoint denoted  $\pi^*$ . We have  $\pi \circ \pi^* \simeq 1$ .*
- (iii) *The functor  $\pi^*$  is fully faithful.*

*Proof.* (i) There is an inclusion functor  $\text{tr}_{n-1}(\vec{r})I^{n-1} \hookrightarrow \vec{r}I^n$  defined by

$$(x_1, x_2, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, 0).$$

The functor  $\pi$  is just the restriction along this inclusion. The exactness follows from the fact that in functor categories, limits and colimits are computed pointwise.

- (ii) Given  $F \in \text{Func}(\text{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$  we need to construct an object  $\pi^*(F) \in \text{Func}(\vec{r}I^n, \mathbf{A})$ . We set

$$\pi^*(F)_{(u_1, u_2, \dots, u_n)} = F_{(u_1, u_2, \dots, u_{n-1})}.$$

To produce a functor, we need maps

$$\lambda_{(u_1, \dots, u_n)}^i : \pi^*(\mathcal{F})_{(u_1, \dots, u_i, \dots, u_n)} \rightarrow \pi^*(F)_{(u_1, \dots, u_i+1, \dots, u_n)}$$

We define

$$\lambda_{(u_1, \dots, u_n)}^i = \begin{cases} F_{(u_1, \dots, u_i, \dots, u_{n-1})} \rightarrow F_{(u_1, \dots, u_i+1, \dots, u_{n-1})} & \text{if } i < n \\ \text{identity} & \text{if } i = n. \end{cases}$$

One checks that the hypothesis of (2.4) are satisfied. Observe that  $\pi \circ \pi^* = 1$ . This produces a natural map

$$\text{Hom}(\pi^*(F), G) \rightarrow \text{Hom}(F, \pi(G)).$$

To see that this is a bijection, suppose that we are given a morphism  $\beta : F \rightarrow \pi(G)$ . There is a diagram, where the dashed arrow is defined to be the composition,

$$\begin{array}{ccc} \pi^*(F)_{(u_1, \dots, u_n)} & \text{-----} \rightarrow & G_{(u_1, \dots, u_n)} \\ \parallel & & \uparrow \\ F_{(u_1, \dots, u_{n-1})} & \xrightarrow{\beta} & G_{(u_1, \dots, u_{n-1}, 0)} \end{array}$$

This produces a natural morphism

$$\text{Hom}(\pi^*(F), G) \leftarrow \text{Hom}(F, \pi(G))$$

and we check that it is inverse to the previous map.

- (iii) We have

$$\text{Hom}(\pi^*(F), \pi^*(F')) = \text{Hom}(F, \pi\pi^*(F')) = \text{Hom}(F, F').$$

□

**Theorem 2.6.** (1) *The functor*

$$\pi : \text{Func}(\vec{r}I^n, \mathbf{A}) \rightarrow \text{Func}(\text{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A})$$

*satisfies the hypothesis of (2.3).*

- (2) *Let  $\vec{s} = (r_1, r_2, \dots, r_{n-1}, r_n - 1)$ . If  $r_n > 0$  then the kernel of this functor is equivalent to  $\text{Func}(\vec{s}I^n, \mathbf{A})$ .*

(3) If  $r_n = 0$  then there is an equivalence of categories

$$\mathrm{Func}(\vec{r}I^n, \mathbf{A}) \cong \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \mathbf{A}).$$

*Proof.* (1) The functor  $\pi$  is exact so it remains to check the two conditions of the theorem. The first condition follows from the fact that  $\pi \circ \pi^*$  is the identity. Now suppose that we have a morphism  $\pi(F) \rightarrow \pi(F')$ . By adjointness we obtain a diagram

$$\begin{array}{ccc} \pi^*\pi(F) & & \\ \downarrow & \searrow & \\ F & & F' \end{array}$$

Applying  $\pi$  to this picture shows that the second condition holds.

(2) The functor  $\pi$  was defined on objects by the rule  $\pi(G)_{(u_1, u_2, \dots, u_{n-1})} = (G)_{(u_1, \dots, u_{n-1}, 0)}$ . So it is clear that if  $\pi G \cong 0$  then  $(G)_{(u_1, \dots, u_{n-1}, 0)} \cong 0$  and to give an object  $G$  of  $\mathbf{ker} \pi$  is the same (up to isomorphism) as giving the objects  $(G)_{(u_1, \dots, u_n)} \in \mathbf{A}$  for all  $u \in \vec{r}I^n, u_n \neq 0$ . And according to Lemma 2.4 it is the same as providing an object of the category  $\mathrm{Func}(\vec{s}I^n, \mathbf{A})$

(3) If  $r_n = 0$  then we have an equivalence of categories  $\mathrm{tr}_{n-1}(\vec{r})I^{n-1} \cong \vec{r}I^n$ . □

### 3. COHERENT SHEAVES ON ROOT STACKS

**3.1. Preliminary results.** Recall that if  $M$  is a commutative monoid then  $\widehat{M} = \mathrm{Hom}(M, \mathbb{G}_m)$  is its dual.

In this subsection we will recall the main constructions and theorems from [BV]. We refer the reader to this paper for further details. Let's start by defining a root stack.

Let  $X$  be a scheme. Denote by  $\mathfrak{Div} X$  the groupoid of line bundles with sections over  $X$ . It has the structure of a symmetric monoidal category with tensor product given by

$$(L, s) \otimes (L', s') = (L \otimes L', s \otimes s').$$

Choosing  $n$  objects  $(L_1, s_1), \dots, (L_n, s_n)$  of  $\mathfrak{Div} X$  allows us to define a symmetric monoidal functor (see [BV, Definition 2.1])

$$\begin{aligned} L : \mathbb{N}^n &\longrightarrow \mathfrak{Div} X \\ (k_1, \dots, k_n) &\longmapsto (L_1, s_1)^{\otimes k_1} \otimes \dots \otimes (L_n, s_n)^{\otimes k_n}. \end{aligned}$$

Such functors arise from morphisms  $X \rightarrow [\mathrm{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$ . Let us recall how.

**Proposition 3.1.** (i) Let  $\mathbf{A}$  be the groupoid whose objects are quasi-coherent  $\mathcal{O}_X$ -algebras  $\mathcal{A}$  with a  $\mathbb{Z}^n = \widehat{\mathbb{N}^n}$ -grading  $\mathcal{A} = \bigoplus_{u \in \mathbb{Z}^n} \mathcal{A}_u$  such that each summand  $\mathcal{A}_u$  is an invertible sheaf. The morphisms are graded algebra isomorphisms. Then there is an equivalence of categories between  $\mathbf{A}^{\text{op}}$  and the groupoid of  $\widehat{\mathbb{N}^n}$ -torsors  $P \rightarrow X$ .  
(ii) Let  $\mathbf{B}$  be the groupoid whose objects are pairs  $(\mathcal{A}, \alpha)$  where  $\mathcal{A}$  is a sheaf of algebra satisfying the conditions in (i) and

$$\alpha : \mathcal{O}_X[\mathbb{N}^n] \rightarrow \mathcal{A}$$

is a morphism respecting the grading. The morphisms in the category  $\mathbf{B}$  are graded algebra morphisms commuting with the structure maps. Then there is an equivalence of categories between  $\mathbf{B}^{\text{op}}$  and the groupoid of morphisms  $X \rightarrow [\text{Spec } \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$

*Proof.* This proposition is a summary of the discussion in [BV, p. 1343-1344], in particular the proof of Proposition 3.25. The detailed proof can be found there. Here we will just illustrate the main idea behind the proof.

(i) The torsor  $\pi : P \rightarrow X$  is determined by the sheaf of algebras  $\pi_*(\mathcal{O}_P)$  which has had a  $\widehat{\mathbb{N}^n}$  action and hence a weight grading. As the torsor is locally trivial, the condition about the summands being invertible follows by considering the algebra associated to the trivial torsor.

(ii) This follows from the standard description of the groupoid of  $X$ -points of a quotient stack. Finally, in [BV], the **fppf** topology is needed but in the present work it is not. The setting in *loc. cit.* is more general and the monoids in question may have torsion so that the torsor  $P$  is a torsor over  $\mu_n$ . Such a torsor may not be trivial in the Zariski topology, unlike a  $\mathbb{G}_m$ -torsor. Hence a finer topology is needed. See the proof of [BV, Lemma 3.26].  $\square$

**Corollary 3.2.** *There is an equivalence of categories between the groupoid of symmetric monoidal functors  $\mathbb{N}^n \rightarrow \mathfrak{Div} X$  and the groupoid of  $X$ -points of  $[\text{Spec } \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$ .*

*Proof.* For details see [BV, Proposition 3.25]. In essence, the symmetric monoidal functor determined by  $(L_1, s_1), \dots, (L_n, s_n)$  produces the graded sheaf of algebras

$$\mathcal{A} = \bigoplus_{\vec{u} \in \mathbb{Z}^n} L_1^{u_1} \otimes \dots \otimes L_n^{u_n}.$$

The sections produce an algebra map

$$\mathcal{O}_X[\mathbb{N}^n] \rightarrow \mathcal{A}.$$

□

**Definition 3.3.** Let  $\vec{r} = (r_1, r_2, \dots, r_n)$  be a collection of positive natural numbers. We denote by  $r_i\mathbb{N}$  the monoid  $\{vr_i | v \in \mathbb{N}\}$ . We denote by  $\vec{r}\mathbb{N}^n$  the monoid

$$\vec{r}\mathbb{N}^n = r_1\mathbb{N} \times r_2\mathbb{N} \times \dots \times r_n\mathbb{N}.$$

We will view our symmetric monoidal functor above as a functor

$$L : \vec{r}\mathbb{N}^n \rightarrow \mathfrak{Div}X$$

in the following way :

$$(r_1\alpha_1, r_2\alpha_2, \dots, r_n\alpha_n) \mapsto (L_1, s_1)^{\otimes \alpha_1} \otimes \dots \otimes (L_n, s_n)^{\otimes \alpha_n}.$$

Consider the natural inclusion of monoids  $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$ . The *category of  $\vec{r}$ th roots of  $L$*  denoted by  $(L)_{\vec{r}}$ , is defined as follows :

Its objects are pairs  $(M, \alpha)$ , where  $M : \mathbb{N}^n \rightarrow \mathfrak{Div}X$  is a symmetric monoidal functor, and  $\alpha : L \rightarrow M \circ j$  is an isomorphism of symmetric monoidal functors.

An arrow from  $(M, \alpha)$  to  $(M', \alpha')$  is an isomorphism  $h : M \rightarrow M'$  of symmetric monoidal functors  $\mathbb{N}^n \rightarrow \mathfrak{Div}X$ , such that the diagram

$$\begin{array}{ccc} & L & \\ \alpha \swarrow & & \searrow \alpha' \\ M \circ j & \xrightarrow{h \circ j} & M' \circ j \end{array}$$

commutes.

This category is in fact a groupoid as a morphism  $\phi$  in  $\mathfrak{Div}X$  whose tensor power  $\phi^{\otimes k}$  is an isomorphism must be an isomorphism to begin with.

Given a morphism of schemes  $t : T \rightarrow X$  there is pullback functor

$$t^* : \mathfrak{Div}X \rightarrow \mathfrak{Div}T.$$

Hence we can form the category of roots  $(t^* \circ L)_{\vec{r}}$ . This construction pastes together to produce a pseudo-functor

$$(\text{Sch}/X) \rightarrow \mathfrak{Div}_X,$$

where  $\mathfrak{Div}_X \rightarrow \text{Sch}/X$  is the symmetric monoidal stack described in [BV, p. 1335].

**Definition 3.4.** In the above situation, the fibred category associated to this pseudo-functor is called *the stack of roots* associated to  $L$  and  $\vec{r}$ . It is denoted by  $X_{L, \vec{r}}$ .



We will often denote the stack of roots by

$$X_{L,\vec{r}} = X_{(L_1,s_1,r_1),\dots,(L_n,s_n,r_n)}.$$

There are also two equivalent definitions of the stack  $X_{L,\vec{r}}$  and the equivalence is proved in [BV, Proposition 4.13] and [BV, Remark 4.14]. Lets recall the description of this stack as a fibered product.

**Proposition 3.5.** *The stack  $X_{L,\vec{r}}$  is isomorphic to the fibered product*

$$X \times_{\mathrm{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} [\mathrm{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$$

According to (a slightly modified version of) Corollary 3.2 a symmetric monoidal functor  $L : \vec{r}\mathbb{N}^n \rightarrow \mathfrak{Div} X$  corresponds to a morphism  $X \rightarrow [\mathrm{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}]$  which in turn corresponds to a  $\widehat{\vec{r}\mathbb{N}^n}$ -torsor  $\pi : P \rightarrow X$  and a  $\vec{r}\mathbb{N}^n$ -equivariant morphism  $P \rightarrow \mathrm{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]$ . This gives

**Proposition 3.6.** *The stack  $X_{L,\vec{r}}$  is isomorphic to the quotient stack*

$$[P \times_{\mathrm{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathrm{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}],$$

where the action on the first factor is defined through the dual of the inclusion  $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$ .

*Proof.* See [BV]. □

We recall the definition of parabolic sheaf, see [BV, Definition 5.6].

**Definition 3.7.** Consider a scheme  $X$ , an inclusion  $\vec{r}\mathbb{N} \subseteq \mathbb{N}$  and the symmetric monoidal functor  $L : \vec{r}\mathbb{N} \rightarrow \mathfrak{Div}_X$  introduced earlier. It is defined by

$$L_u = L(u) = L_1^{\alpha_1} \otimes \dots \otimes L_n^{\alpha_n},$$

where  $u = (r_1\alpha_1, \dots, r_n\alpha_n)$ . A *parabolic sheaf*  $(E, \rho)$  on  $(X, L)$  with denominators  $\vec{r}$  consists of the following data :

- (a) A functor  $E : \mathbb{Z}^n \rightarrow \mathfrak{Qcoh} X$ , denoted by  $v \mapsto E_v$  on objects and  $b \mapsto E_b$  on arrows.
- (b) For any  $u \in \vec{r}\mathbb{Z}^n$  and  $v \in \mathbb{Z}^n$ , an isomorphism of  $\mathcal{O}_X$ -modules:

$$\rho_{u,v}^E : E_{u+v} \simeq L_u \otimes_{\mathcal{O}_X} E_v.$$

This map is called the *pseudo-period isomorphism*.

This data are required to satisfy the following conditions. Let  $u, u' \in \vec{r}\mathbb{Z}^n$ ,  $a = (r_1\alpha_1, \dots, r_n\alpha_n) \in \vec{r}\mathbb{N}^n$ ,  $b \in \mathbb{N}^n$ ,  $v \in \mathbb{Z}^n$ . Then the following diagrams commute.

(i)

$$\begin{array}{ccc}
E_v & \xrightarrow{E_a} & E_{a+v} \\
\downarrow \simeq & & \downarrow \rho_{a,v}^E \\
\mathcal{O}_X \otimes E_v & \xrightarrow{\sigma_a^L \otimes id_{E_v}} & L_a \otimes E_v
\end{array}$$

where  $\sigma_a = \sigma^{\alpha_1} \otimes \dots \otimes \sigma^{\alpha_n} \in H^0(X, L_a)$

(ii)

$$\begin{array}{ccc}
E_{u+v} & \xrightarrow{\rho_{u,v}^E} & L_u \otimes E_v \\
\downarrow E_b & & \downarrow id \otimes E_b \\
E_{u+b+v} & \xrightarrow{\rho_{u,b+v}^E} & L_u \otimes E_{b+v}
\end{array}$$

(iii)

$$\begin{array}{ccc}
E_{u+u'+v} & \xrightarrow{\rho_{u+u',v}^E} & L_{u+u'} \otimes E_v \\
\downarrow \rho_{u,u'+v}^E & & \downarrow \mu \otimes id \\
L(u) \otimes E_{u'+v} & \xrightarrow{id \otimes \rho_{u',v}^E} & L_u \otimes L_{u'} \otimes E_v
\end{array}$$

(iv) The map

$$E_v = E_{0+v} \xrightarrow{\rho_{0,v}^E} \mathcal{O}_X \otimes E_v$$

is the natural isomorphism.

**Definition 3.8.** A parabolic sheaf  $(E, \rho)$  is said to be *coherent* if for each  $v \in \mathbb{Z}^n$  the sheaf  $E_v$  is a coherent sheaf on  $X$ .

**Theorem 3.9** (Borne, Vistoli). *Let  $X$  be a scheme and  $L$  is a monoidal functor defined as in the beginning of the subsection. Then there is a canonical tensor equivalence of abelian categories between the category  $\mathfrak{QCoh}_{X_{L,\vec{r}}}$  and the category of parabolic sheaves on  $X$ , associated with  $L$  with denominator  $\vec{r}$ .*

*Proof.* See [BV, Proposition 5.10, Theorem 6.1] for details. The proof relies on the description of the stack as a quotient, (3.6). From this description, sheaves on the stack are equivariant sheaves on

$$P \times_{\mathrm{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathrm{Spec} \mathbb{Z}[\mathbb{N}^n].$$

As remarked in the proof of (3.1), the torsor  $P$  is obtained from a sheaf of algebras on  $X$ . The sheaf of algebras  $\mathcal{A}$  is constructed from the functor  $L$  by taking a direct sum construction, it has a natural grading. It follows that the scheme

$$P \times_{\mathrm{Spec} \mathbb{Z}[\tilde{r}\mathbb{N}^n]} \mathrm{Spec} \mathbb{Z}[\mathbb{N}^n] = \mathrm{Spec}(\mathcal{A} \otimes_{\mathbb{Z}[\tilde{r}\mathbb{N}^n]} \mathbb{Z}[\mathbb{N}^n]).$$

The algebra on the right has a natural  $\mathbb{Z}[\mathbb{N}^n]$  grading, see the corollary below for a local description. It follows that the equivariant sheaves on the scheme in question are just graded modules over this algebra. The proof follows by reinterpreting the graded modules in terms of the symmetric monoidal functor  $L$ .  $\square$

Actually we can add the finiteness condition to the previous theorem and get the following

**Corollary 3.10.** *There is a canonical tensor equivalence of abelian categories between the category  $\mathfrak{Coh} X_{L,\tilde{r}}$  and the category of coherent parabolic sheaves on  $X$ , associated with  $L$ .*

*Proof.* We will make use of the identifications in the above proof. Further the question is local on  $X$ , so we may assume that  $X$  is in fact an affine scheme  $\mathrm{Spec}(R)$ . By further restrictions we can assume that all the line bundles  $L_i$  are in fact trivial, and we identify them with  $R$ . In this situation the symmetric monoidal functor corresponds to a graded homomorphism

$$\mathbb{Z}[X_1, X_2, \dots, X_n] \rightarrow R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$$

sending  $X_i$  to  $x_i t_i$  with  $x_i \in R$ . Further the morphism

$$\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^n]) \rightarrow \mathrm{Spec}(\mathbb{Z}[\tilde{r}\mathbb{N}^n])$$

comes from an integral extension of algebras

$$\mathbb{Z}[X_1, X_2, \dots, X_n][Y_1, \dots, Y_n] / (Y_1^{r_1} - X_1, \dots, Y_n^{r_n} - X_n)$$

Then taking tensor products yields a  $\mathbb{Z}^n$ -graded algebra

$$A = R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}][s_1, \dots, s_n] / (s_1^{r_1} - x_1 t_1, \dots, s_n^{r_n} - x_n t_n)$$

where  $s_i$  has degree  $(0, \dots, 0, 1, 0, \dots, 0) = e_i$ . Consider now  $M$  a finitely generated graded  $A$ -module. We can assume that the generators of  $M$  are in fact homogeneous and hence there is an epimorphism

$$\bigoplus_{i=1}^p A(n_i) \rightarrow M.$$

The graded pieces of the module on the left are free of rank  $p$  and hence the graded pieces of  $M$  are finitely generated. It follows that a finitely generated  $A$ -module gives rise to a parabolic sheaf with values

in the category of finitely generated  $R$ -modules, in other words coherent sheaves on  $X$ .

Conversely suppose that each we have a graded  $A$ -module  $M$  with each graded piece a finitely generated  $R$ -module. We can find finitely many elements of  $M$ , lets say  $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$  of degrees

$$\deg(\alpha_i) = (\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in}) \in \mathbb{Z}^n$$

with  $0 \leq \lambda_{ij} \leq r_j$  such that the associated morphism

$$\phi : \bigoplus_{i=1}^p A(\deg(\alpha_i)) \rightarrow M$$

is an epimorphism in degrees

$$(\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$$

whenever  $0 \leq \mu_i \leq r_i$ . It follows that  $\phi$  is an epimorphism at multiplication by  $t_i$  induces an isomorphism  $M_v \xrightarrow{\sim} M_{v+e_i}$ .  $\square$

**3.2. An extension lemma.** The goal of this subsection is to slightly simplify the formulation of parabolic sheaf in the present context using the pseudo-periodicity condition. This will be needed to study  $K$ -theory in the next section. We let

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n,$$

where the 1 is in the  $i$ th spot.

**Definition 3.11.** Recall that  $L$  is a symmetric monoidal functor

$$L : \vec{r}\mathbb{Z} \rightarrow \mathfrak{Div}_X,$$

determined by  $n$  divisors  $(L_i, s_i)$ . An *extendable pair*  $(F, \rho)$  on  $(X, L)$  consists of the following data:

- (a) A functor  $F_\bullet : \vec{r}I^n \rightarrow \mathfrak{QCoH}(X)$ .
- (b) For any  $\alpha \in \vec{r}I^n$  such that  $\alpha_i = r_i$ , an isomorphism of  $\mathcal{O}_X$ -modules

$$\rho_{\alpha, \alpha - r_i e_i} : F_\alpha \xrightarrow{\sim} L_i \otimes F_{\alpha - r_i e_i}.$$

We will frequently drop the subscripts from the notation involving  $\rho$ , when they are clear from the context.

This data is required to satisfy the following three conditions :

- (EX1) For all  $i \in \{1, \dots, n\}$  and  $\alpha \in \vec{r}I^n$  the following diagram commutes

$$\begin{array}{ccc}
F_\alpha & \xrightarrow{F_{(r_i - \alpha_i)e_i}} & F_{\alpha + (r_i - \alpha_i)e_i} \\
\downarrow \sigma_i & & \downarrow \rho \\
L_i \otimes F_\alpha & \xleftarrow{L_i \otimes F_{\alpha_i e_i}} & L_i \otimes F_{\alpha - \alpha_i e_i}
\end{array}$$

where  $\sigma_i$  is multiplication by the section  $s_i$

(EX2) For all  $i \neq j$  and  $\alpha$  with  $\alpha_i = r_i$  the following diagram commutes

$$\begin{array}{ccc}
F_\alpha & \xrightarrow{\rho} & L_i \otimes F_{\alpha - r_i e_i} \\
\downarrow F_{e_j} & & \downarrow F_{e_j} \\
F_{\alpha + e_j} & \xrightarrow{\rho} & L_i \otimes F_{\alpha + e_j - r_i e_i}
\end{array}$$

(EX3) For all  $i$  and  $j$  and  $\alpha \in \vec{r}I^n$  with  $\alpha_i = r_i$  and  $\alpha_j = r_j$  the following diagram commutes

$$\begin{array}{ccc}
F_\alpha & \xrightarrow{\rho} & L_i \otimes F_{\alpha - r_i e_i} \\
\downarrow \rho & & \downarrow \rho \\
L_j \otimes F_{\alpha - r_j e_j} & \xrightarrow{\rho} & L_i \otimes L_j \otimes F_{\alpha - r_i e_i - r_j e_j}
\end{array}$$

**Definition 3.12.** An extendable pair  $(F, \rho)$  is called *coherent* if for each  $v \in \vec{r}I^n$  the sheaf  $F_v$  is a coherent sheaf on  $X$ .

**Proposition 3.13.** Let  $(E, \rho)$  be a parabolic sheaf on  $(X, L)$  with denominators  $\vec{r}$ . Then the restricted functor  $E|_{\vec{r}I^n}$  produces an extendable pair on  $(X, L)$ .

*Proof.* Note that the restricted functor has all the required data for an extendable pair by restricting the collection  $\rho_{\alpha, \beta}$ . We need to check that the axioms of an extendable pair are satisfied.

(EX 1) We have that the composition

$$E_{\alpha + (r_i - \alpha_i)e_i} \xrightarrow{\rho} E_{\alpha - \alpha_i e_i} \otimes L_i \rightarrow E_\alpha \otimes L_i \xrightarrow{\rho^{-1}} E_{\alpha + r_i e_i}$$

is just the morphism  $E_{\alpha_i e_i}$  using axiom (ii) of parabolic sheaf. Precomposing with the map

$$E_{(r_i - \alpha_i)e_i} : E_\alpha \rightarrow E_{\alpha + (r_i - \alpha_i)e_i}$$

gives the morphism  $E_{r_i e_i}$ . The result now follows from axiom (i).

(EX 2) This follows directly from axiom (ii).

(EX 3) This follows directly from axiom (iii).  $\square$

**Proposition 3.14.** *Given an extendable pair  $(F, \rho)$  we can extend it to a parabolic sheaf  $(\hat{F}, \rho)$  and the extension is unique up to a canonical isomorphism. A coherent extendable pair extends to a coherent parabolic sheaf.*

*Proof.* For  $v \in \mathbb{Z}^n$  we need to define its extension  $\hat{F}_v$ . We can write  $v_i = r_i u_i + q_i$  with  $0 \leq q_i < r_i$  and  $u_i \in \mathbb{Z}$ . As before we denote  $L_u = \bigotimes_{i=1}^n L^{\otimes u_i}$  and  $q = (q_1, \dots, q_n)$ . Set  $\hat{F}_v = L_u \otimes F_q$ .

We need to construct maps  $\hat{F}_{+e_i} : \hat{F}_v \rightarrow \hat{F}_{v+e_i}$ . If  $q_i < r_i - 1$  then the map is obtained by tensoring the map  $F_{q_i} \rightarrow F_{q_i+e_i}$  with  $L_u$ . If  $q_i = r_i - 1$  then the map is defined by

$$\begin{array}{ccc} \hat{F}_v = L_u \otimes F_q & \xrightarrow{\hat{F}_{e_i}} & \hat{F}_{v+e_i} = L_u \otimes L_i \otimes F_{q'} \\ & \searrow 1 \otimes F_{e_i} \quad \nearrow 1 \otimes \rho & \\ & L_u \otimes F_{q+e_i} & \end{array}$$

where  $q'_j = q_j$  for all  $j \neq i$  and  $q'_i = 0$ .

In order to show that the construction above indeed produces a functor we need to show that all diagrams of (2.4) commute. If both  $q_i < r_i - 1$  and  $q_j < r_j - 1$  then this is straightforward. Suppose that  $q_i = r_i - 1$  and  $q_j < r_j - 1$  then this follows from (EX2). This leaves the case  $q_i = r_i - 1$  and  $q_j = r_j - 1$ . We have a diagram

$$\begin{array}{ccccc}
 L_u \otimes F_q & \longrightarrow & L_u \otimes F_{q+e_i} & \longrightarrow & L_u \otimes L_i \otimes F_{q-q_i e_i} \\
 \downarrow & & \downarrow & & \downarrow \\
 L_u \otimes F_{q+e_j} & \longrightarrow & L_u \otimes F_{q+e_i+e_j} & \longrightarrow & L_u \otimes L_i \otimes F_{q-q_i e_i+e_j} \\
 \downarrow & & \downarrow & & \downarrow \\
 L_u \otimes L_j \otimes F_{q-q_j e_j} & \longrightarrow & L_u \otimes L_j \otimes F_{q-q_j e_j+e_i} & \longrightarrow & L_u \otimes L_i \otimes L_j \otimes F_{q-q_i e_i-q_j e_j}
 \end{array}$$

The top left square commutes using the fact that  $F$  is a functor. The top right and bottom left squares commute using axiom (EX2). The bottom right square commutes using axiom (EX3). So indeed  $\hat{F}_\bullet$  is a functor.

Note that we have canonical isomorphisms  $L_u \otimes L_v \cong L_{u+v}$  for  $u, v \in \bar{r}\mathbb{Z}$ . These isomorphisms induce our pseudo-period isomorphisms.

Finally we need to check the conditions (i) to (iv) of a parabolic sheaf.

Condition (i): For  $\vec{r}\alpha, \vec{r}\alpha' \in \bar{r}\mathbb{N}^n$  the following diagram commutes

$$\begin{array}{ccccc}
 \hat{F}_v & \xrightarrow{\hat{F}_{\vec{r}\alpha}} & \hat{F}_{v+\vec{r}\alpha} & \xrightarrow{\hat{F}_{\vec{r}\alpha'}} & \hat{F}_{v+\vec{r}\alpha+\vec{r}\alpha'} \\
 & & \downarrow & & \downarrow \\
 & & L_\alpha \otimes \hat{F}_v & \longrightarrow & L_\alpha \otimes L_{\alpha'} \otimes \hat{F}_v
 \end{array}$$

This follows by the definition of the functor  $\hat{F}_\bullet$  and the symmetric monoidal structure of  $L$ .

This allows us to make the following reduction: in order to check axiom (i) it suffices to check that the following diagram commutes

$$\begin{array}{ccc}
 \hat{F}_v & \longrightarrow & \hat{F}_{v+r_i e_i} \\
 \searrow \sigma_i & & \downarrow \rho \\
 & & L_i \otimes \hat{F}_v
 \end{array}$$

And this follows directly from (EX1).

Condition (ii): Once again we reduce to showing that

$$\begin{array}{ccc}
\hat{F}_{v+r_i e_i} & \longrightarrow & L_i \otimes \hat{F}_v \\
\downarrow \hat{F}_b & & \downarrow L_i \otimes \hat{F}_b \\
\hat{F}_{v+b+r_i e_i} & \longrightarrow & L_i \otimes \hat{F}_{v+b}
\end{array}$$

commutes. If we write  $v = \vec{r}u + q$  then this diagram will become

$$\begin{array}{ccc}
L_{u+e_i} \otimes F_q & \longrightarrow & L_i \otimes (L_u \otimes F_q) \\
\downarrow L_{u+e_i} \otimes \hat{F}_b & & \downarrow L_i \otimes L_u \otimes \hat{F}_b \\
L_{u+e_i} \otimes \hat{F}_{q+b} & \longrightarrow & L_i \otimes (L_u \otimes \hat{F}_{q+b})
\end{array}$$

We can use the symmetric monoidal structure of  $L$  to show that this diagram indeed commutes.

Condition (iii): We reduce to showing the commutativity of the following diagram

$$\begin{array}{ccc}
\hat{F}_{v+r_i e_i + r_j e_j} & \longrightarrow & L_i \otimes \hat{F}_{v+r_j e_j} \\
\downarrow & & \downarrow \\
L_j \otimes \hat{F}_{v+r_i e_i} & \longrightarrow & L_i \otimes L_j \otimes \hat{F}_v
\end{array}$$

which follows from the monoidal structure of  $L$ .

Condition (iv) is by definition.

Finally, let  $E_\bullet$  be another extension of  $F_\bullet$ . Again we can again write  $v_i = r_i u_i + q_i$  with  $0 \leq q_i < r_i$  and  $u_i \in \mathbb{Z}$ . By pseudo-periodicity,  $E_v \simeq L(u) \otimes E_q$ , and  $F_q = E_q$  because  $E_\bullet$  is an extension. So,  $E_v \cong \hat{F}_v$  for any  $v \in \mathbb{Z}^n$ .

It is clear from the construction that finitely generated condition is preserved under extension.  $\square$

**Corollary 3.15.** *The category of parabolic sheaves (coherent parabolic sheaves) on  $(X, L)$  is equivalent to the category of extendable pairs (resp. coherent extendable pairs) on  $(X, L)$ .*

*Proof.* There is a pair of functors between these categories. The truncation functor sends a parabolic sheaf  $(E, \rho)$  to an extendable pair by



forgetting all  $E_v$  when  $v \notin \vec{r}I^n$ . And the extension functor from extendable pairs to parabolic sheaves was defined in the previous Proposition on objects by the rule  $F_\bullet \mapsto \hat{F}_\bullet$ . It is easy to see that these functors are mutually inverse and preserve finitely-generation condition.  $\square$

**Remark 3.16.** We will denote the category of coherent extendable pairs by  $\mathcal{EP}(X, L, \vec{r})$ .

**3.3. The localization sequence.** In this subsection we will localize the category of finitely-generated extendable pairs so that it will be glued from simpler parts.

First let us consider the functor  $\pi_*^{L, \vec{r}} : \mathcal{EP}(X, L, \vec{r}) \rightarrow \mathfrak{Coh}X$ , given by  $F_\bullet \mapsto F_0$  on objects. It is an exact functor because exact sequences in diagram categories are defined point-wise.

**Lemma 3.17.** *The functor  $\pi_*^{L, \vec{r}}$  has a left adjoint denoted  $\pi_{L, \vec{r}}^*$  and there is a natural isomorphism  $\pi_*^{L, \vec{r}} \circ \pi_{L, \vec{r}}^* \simeq 1$ .*

*Proof.* In what follows, we will omit, the superscripts (resp. subscripts)  $L$  and  $\vec{r}$  in the notation for the appropriate functors. For any  $0 \leq i \leq n$  consider functions  $\epsilon_i : \vec{r}I \rightarrow \{0, 1\}$ , defined by  $\epsilon_i(u) = 1$  if  $u_i = r_i$  and zero otherwise. We define the functor  $\pi^*$  on a sheaf  $F \in \mathfrak{Coh}X$  by the rule:

$$(\pi^*(F))_u = (\otimes_{i=1}^n L_i^{\epsilon_i(u)}) \otimes F.$$

This forms a functor via the maps

$$(\pi^*(F))_u \rightarrow (\pi^*(F))_{u+e_i} = \begin{cases} \text{identity} & \text{if } u_i \in [0, r_i - 2] \\ \sigma_i & \text{if } u_i = r_i - 1, \end{cases}$$

where  $\sigma_i$  is the multiplication by the section  $s_i$ .

Define  $\rho$  to be identity map. It is easy to see that all axioms of extendable pair are satisfied.

Now let's take a coherent sheaf  $F$  and an extendable pair  $E_\bullet$  and consider a map

$$\text{Hom}_{\mathfrak{Coh}X}(F, \pi_* E) \rightarrow \text{Hom}_{\mathcal{EP}}(\pi^* F, E)$$

given by sending  $\phi \in \text{Hom}_{\mathfrak{Coh}X}(F, \pi_* E)$  to precomposition of the structure maps of the extendable pair  $E$  with  $\phi$ . It's obviously an injection. Surjectivity will follow from commutativity of the squares in  $\text{Hom}_{\mathcal{EP}}(\pi^* F, E)$  and because all structure maps in  $\pi^* F$  are identity.  $\square$

**Proposition 3.18.** *The functor  $\pi_*^{L, \vec{r}} : \mathcal{EP}(X, L, \vec{r}) \rightarrow \mathfrak{Coh}X$  satisfies the hypothesis of (2.3).*

*Proof.* The only thing which is not completely obvious is the second condition. Consider two extendable pairs  $E_\bullet$  and  $F_\bullet$ . Suppose that we have a morphism  $\pi_*(E_\bullet) \rightarrow \pi_*(F_\bullet)$ . By adjointness we obtain a diagram

$$\begin{array}{ccc} \pi^*\pi_*(E_\bullet) & & \\ \downarrow & \searrow & \\ E_\bullet & & F_\bullet \end{array}$$

Applying  $\pi$  to this picture shows that the second condition holds.  $\square$

Using the Theorem 2.3 we obtain the following

**Corollary 3.19.** *There is an equivalence of abelian categories:*

$$\mathcal{EP}(X, L, \vec{r}) / \mathbf{ker}(\pi_*^{L, \vec{r}}) \rightarrow \mathfrak{Coh} X,$$

In the rest of this subsection we would like to give a description of the category  $\mathbf{ker}(\pi_*^{L, \vec{r}})$ . Let us study the objects first. Let  $F_\bullet$  be an extendable pair. Then  $\pi_*(F_\bullet) = F_0$ , and if  $F_\bullet \in \mathbf{ker}(\pi_*^{L, \vec{r}})$  then  $F_0 \cong 0$ . The pseudo-period isomorphism imply in turn that  $F_u \cong 0$  if all  $u_i \in \{0, r_i\}$ .

Let us consider the sheaves  $F_u$  such that for any  $j \neq i$   $u_j \in \{0, r_j\}$  (we can imagine them as sheaves on the edges of the cubical diagram  $F_\bullet \in \text{Func}(\vec{r}I^n, \mathbf{A})$ ). Using the axiom (EX 1) we get that the multiplication by section map  $s_i : F_u \rightarrow L_i \otimes F_u$  must factor through  $F_{u+(r_i-u_i)e_i}$  which is a zero sheaf if  $F_\bullet \in \mathbf{ker}(\pi_*^{L, \vec{r}})$ . This implies the following:

**Lemma 3.20.** *If  $F_\bullet \in \mathbf{ker}(\pi_*^{L, \vec{r}})$  and  $u \in \vec{r}I^n$  is such that  $\forall j \neq i$   $u_j \in \{0, r_j\}$  then  $\text{supp}(F_u)$  is contained in the divisor of zeroes of the section  $s_i \in H^0(L_i)$ .*

*If  $s_i = 0$  for some  $i$ , we will say that  $\text{div}(s_i) = X$ .*

We will apply the localization method (2.3), to this partial description of the kernel.

Lets fix some notation. Denote by

$$S(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}.$$

We will view each interval  $[0, r_i]$  as a pointed set, pointed at 0. It follows that we have order preserving inclusions

$$\iota_T : \prod_{i \in T} [0, r_i] \rightarrow \prod_{i=1}^n [0, r_i] := \vec{r}I^n.$$

Ignoring the pointed structure produces order preserving ( $\leq$ ) projection maps

$$\pi_T : \vec{r}I^n \rightarrow \prod_{i \in T} [0, r_i].$$

**Definition 3.21.** Assume that  $(L_1, s_1), \dots, (L_n, s_n)$  are objects of  $\mathfrak{Div}X$  and  $L : \mathbb{N}^n \rightarrow \mathfrak{Div}X$  is the corresponding symmetric monoidal functor as in section 3.1.

If  $1 \leq k \leq n$  and  $T \in S(k)$  then we will define a symmetric monoidal functor  $L_T : \mathbb{N}^k \rightarrow \mathfrak{Div}X$  as a composition:

$$\mathbb{N}^k \xrightarrow{\iota_T} \mathbb{N}^n \xrightarrow{L} \mathfrak{Div}X$$

We will say that  $L_T$  is obtained from  $L$  by the *restriction along*  $\iota_T$ .

Now for  $T \in S(k)$  let's consider the functor

$$\iota_T^* : \mathcal{EP}(X, L, \vec{r}) \longrightarrow \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$$

which is the restriction of an extendable pair  $F_\bullet$  along the inclusion  $\iota_T$ . The pseudo-period isomorphism is just obtained by restriction.

**Definition 3.22.** For any  $1 \leq k \leq n$  we define functors

$$\text{Face}^k := \prod_{T \in S(k)} \iota_T^* : \mathcal{EP}(X, L, \vec{r}) \longrightarrow \prod_{T \in S(k)} \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$$

**Definition 3.23.** For any  $1 \leq k \leq n$  we denote by  $\mathbf{ker}^k = \mathbf{ker}(\text{Face}^k)$ .

Also denote  $\mathbf{ker}^0 = \mathbf{ker}(\pi_*)$ .

**Lemma 3.24.** For any  $1 \leq k \leq n$ , any  $F_\bullet \in \mathbf{ker}^{k-1}$  and any  $T \in S(k)$  we can consider  $(\iota_T^*(F_\bullet))_\bullet$  as an element of

$$\text{Func}\left(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\left(\bigcap_{i \in T} \text{div}(s_i)\right)\right).$$

As in Lemma 3.20 we will say that if  $s_i = 0$ , then  $\text{div}(s_i) = X$ .

*Proof.* If  $k = 1$  then the result is proved in the Lemma 3.20 and the observation before it.

Let's take any  $2 \leq k \leq n$  and an extendable pair  $F_\bullet \in \mathbf{ker}^{k-1}$ .

If we consider an extendable pair  $(\iota_T^*(F_\bullet))_\bullet \in \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$  then for any  $v \in \prod_{i \in T} [0, r_i]$  we will have isomorphisms of sheaves:  $(\iota_T^*(F_\bullet))_v \cong 0$ , whenever  $v_i = 0$  for some  $i \in T$ . Because of the pseudo-periodicity isomorphism we also have that  $(\iota_T(F_\bullet))_v \cong 0$ , whenever  $v_i = r_i$  for some  $i \in T$ .

The last step is an application of the axiom EX1 to the extendable pair  $(\iota_T^*(F_\bullet))_\bullet$ . Because  $(\iota_T^*(F_\bullet))_v \cong 0$  if  $v_i = r_i$  for some  $i \in T$  that

implies that for any  $w \in \prod_{i \in T} [1, r_i - 1]$  the multiplication of the sheaf  $(\iota_T^*(F_\bullet))_w$  by the sections  $s_i \in H^0(X, L_i)$  for all  $i \in T$  must factor through zero. So the support of the sheaf  $(\iota_T^*(F_\bullet))_w$  is contained in  $\bigcap_{i \in T} \text{div}(s_i)$ .

□

**Lemma 3.25.** *If we restrict the domain of the functor  $\text{Face}^k$  to the full subcategory  $\mathbf{ker}^{k-1}$  for any  $1 \leq k \leq n$ , then we will obtain functors:*

$$\text{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \longrightarrow \prod_{T \in S(k)} \text{Func}\left(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\left(\bigcap_{i \in T} \text{div}(s_i)\right)\right).$$

*There is an equivalence of categories  $\mathbf{ker}^k$  and  $\mathbf{ker}(\text{Face}^k|_{\mathbf{ker}^{k-1}})$ .*

*Proof.* The first part follows directly from the Lemma before. The proof of the second part is straightforward and follows from the fact that  $\mathbf{ker}^k$  is a full subcategory of  $\mathbf{ker}^{k-1}$ . □

**Remark 3.26.** In order to apply localization procedure to the category  $\mathbf{ker}^{k-1}$  we need to show that the functor  $\text{Face}^k|_{\mathbf{ker}^{k-1}}$  has a left adjoint. The existence of a left adjoint follows from special adjoint functor theorem. But for the purpose of splitting of the corresponding short exact sequence of  $K$ -groups (see the section 3.4 for details) we need the unit of the adjunction to be the natural isomorphism. This doesn't follow from the abstract nonsense, so we need an explicit construction of a left adjoint functor. It is given in the following theorem.

**Theorem 3.27.** (i) *For any  $1 \leq k \leq n$  there is an exact functor*

$$\text{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \longrightarrow \prod_{T \in S(k)} \text{Func}\left(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\left(\bigcap_{i \in T} \text{div}(s_i)\right)\right),$$

*where  $\mathbf{ker}^k$  is a kernel of the functor  $\text{Face}^k$  and  $\mathbf{ker}^0 := \mathbf{ker}(\pi_*^{L, \vec{r}})$*

(ii) *The functors  $\text{Face}^k|_{\mathbf{ker}^{k-1}}$  have left adjoints  $D^k$  such that*

$$\text{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k \simeq 1$$

(iii)  *$\text{Face}^k|_{\mathbf{ker}^{k-1}}$  satisfies the condition of 2.3*

(iv) *The functor*

$$\text{Face}^n|_{\mathbf{ker}^{n-1}} : \mathbf{ker}^{n-1} \longrightarrow \text{Func}\left(\prod_{i=1}^n [1, r_i - 1], \mathfrak{Coh}\left(\bigcap_{i=1}^n \text{div}(s_i)\right)\right)$$

*is an equivalence of categories.*

*Proof.* (i) This follows from the fact that restriction or pullback functors are exact in general.

(ii) Given a functor  $G_{\bullet}^T \in \text{Func}(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}(\bigcap_{i \in T} \text{div}(s_i)))$  for each  $T \in S(k)$ , we will denote the corresponding object:

$$(G_{\bullet}^T)_{T \in S(k)} \in \prod_{T \in S(k)} \text{Func}(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}(\bigcap_{i \in T} \text{div}(s_i))).$$

Further we will view  $G_{\bullet}^T$  as a functor  $\prod_{i \in T} [0, r_i] \rightarrow \mathfrak{Coh}(\bigcap_{i \in T} \text{div}(s_i))$  by taking  $G_u^T = 0$  if for some  $i \in T$  we have  $u_i \in \{0, r_i\}$ , where 0 is some fixed zero object in  $\mathfrak{Coh}(X)$ . Also for  $i \in \{1, \dots, k\}$  if  $u_i \in \{0, r_i - 1\}$  we define the morphisms  $G_{e_i}^T : G_u^T \rightarrow G_{u+e_i}^T$  as the initial and terminal map correspondingly.

Let's remind the definition of  $\epsilon$  from the Lemma 3.17. For any  $0 \leq i \leq n$  we have functions  $\epsilon_i : \bar{r}I \rightarrow \{0, 1\}$ , such that for any  $u \in \bar{r}I^n$ :  $\epsilon_i(u) = 1$  if  $u_i = r_i$  and zero otherwise.

We define the functor  $D^k$  on objects as follows:

$$(D^k((G_{\bullet}^T)_{T \in S(k)}))_u = (\otimes_{i=1}^n L_i^{\epsilon_i(u)}) \otimes (\oplus_{T \in S(k)} G_{\pi_T(u)}^T)$$

Let's denote  $(D^k((G_{\bullet}^T)_{T \in S(k)}))_{\bullet}$  by  $D_{\bullet}^k$  for the simplicity of notations. First of all we want to view it as a functor  $\bar{r}I^n \rightarrow \mathfrak{Coh}(X)$ . For that we have to define the morphisms:

$$D_{e_i}^k : D_u^k \rightarrow D_{u+e_i}^k.$$

If  $0 \leq u_i < r_i - 1$ , then this map is induced by  $\oplus_{\substack{T \in S(k) \\ \text{s.t. } i \in T}} G_{+1}^T$ . If  $u_i = r_i - 1$ , then it is induced by the terminal maps  $\oplus_{\substack{T \in S(k) \\ \text{s.t. } i \in T}} G_{+1}^T$ , and also by multiplication by the section  $s_i$ .

The pseudo-period isomorphisms  $\rho$  are defined by the symmetric monoidal structure of the functor  $L$ . The proof of the axioms EX2 and EX3 is automatic. And the proof of EX1 will follow from the commutativity of the diagram:

$$\begin{array}{ccc} D_u & \xrightarrow{D_{(r_i-u_i)e_i}} & D_{u+(r_i-u_i)e_i} \\ \downarrow \sigma_i & & \downarrow \rho \\ L_i \otimes D_u & \xleftarrow{L_i \otimes D_{u_i e_i}} & L_i \otimes D_{u-u_i e_i} \end{array}$$

This diagram will commute because of the definition of  $D_{(r_i - \alpha_i)\vec{e}_i}$  and because  $\text{supp}(G_u^T) \subseteq \bigcap_{i \in T} \text{div}(s_i)$  for any  $u \in \prod_{i \in T} [0, r_i]$ .

So we have shown that  $D_\bullet^k$  is an extendable pair. If  $k = 1$  then it's clear that  $D_\bullet^1$  is in  $\mathbf{ker}^0$ , because  $D_0^1 \cong 0$ .

If  $2 \leq k \leq n$ , we want to see that  $D_\bullet^k$  is in  $\mathbf{ker}^{k-1}$ . For that we have to see that for any  $W \in S(k-1)$  and any  $v \in \prod_{i \in W} [0, r_i]$  the sheaf  $(\iota_W^*(D_\bullet^k))_v$  is isomorphic to zero. But this is true because for any  $T \in S(k)$  we have that  $G_u^T = 0$  if  $u_i \in \{0, r_i\}$  for some  $i \in T$ .

Clearly,  $\text{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k = 1$ .

Next we would like to show that  $D^k$  is indeed a left adjoint. Suppose that we have a morphism

$$(G_\bullet^T)_{T \in S(k)} \rightarrow \text{Face}^k(F_\bullet).$$

Such a morphism consists of an  $\binom{n}{k}$ -tuple of morphisms

$$\phi_T : G_\bullet^T \rightarrow \iota_T^*(F_\bullet).$$

We wish to describe the adjoint map

$$\tilde{\phi} : D_\bullet^k \rightarrow F_\bullet.$$

Using the universal property of coproduct, this morphism is determined by maps

$$\tilde{\phi}(u)_T : \otimes_{i=1}^n L_i^{\epsilon(u)} \otimes G_{\pi_T(u)}^T \rightarrow F_u.$$

If  $u$  is such that  $\epsilon_i(u) = 0$  for all  $1 \leq i \leq n$ , then these maps are just the compositions of  $\phi_T$  with the morphisms  $F_{+\alpha}$ . If there are  $l$ 's, such that  $u_l = r_l$ , then  $\tilde{\phi}(u)_T$  is induced by the composition of  $\phi_T$  with  $\rho_F^{-1}$  and with  $F_{+\alpha}$ .

We want to check that the map  $\tilde{\phi}$  is indeed a natural transformation of functors. It's enough to check that the diagram commutes:

$$\begin{array}{ccc} D_u & \xrightarrow{\tilde{\phi}(u)} & F_u \\ \downarrow D_{e_i} & & \downarrow F_{e_i} \\ D_{u+e_i} & \xrightarrow{\tilde{\phi}(u+e_i)} & F_{u+\vec{e}_i} \end{array}$$

If  $\epsilon_k(u) = 0$  for all  $1 \leq k \leq n$  and also  $u_i < r_i - 1$ , then it commutes directly from the construction of the maps  $\tilde{\phi}(u)$ . Otherwise the commutativity will follow from EX1, EX2 and EX3 for  $F_\bullet$ .

Finally we obtained the map:

$$\mathrm{Hom}((G_{\bullet}^T)_{T \in S(k)}, \mathrm{Face}^k(F_{\bullet})) \rightarrow \mathrm{Hom}(D^k((G_{\bullet}^T)_{T \in S(k)}), F_{\bullet}).$$

It's easy to see that this map is bijective, because the right Hom is uniquely defined by the restriction to  $k$ -faces.

(iii) Follows from (ii).

(iv) Because for  $S(n)$  there is only one element, the set  $\{1, \dots, n\}$  itself, we have that  $\iota_{\{1, \dots, n\}} = id$  and  $\pi_{\{1, \dots, n\}} = id$ . So  $\mathrm{Face}|_{\mathbf{ker}^{n-1}}^n$  and  $D^n$  are identity functors.

□

**3.4.  $G$ -theory and  $K$ -theory of a root stack.** In this subsection we will finally describe the  $G$ -theory of a root stack  $X_{L, \vec{r}}$ .

According to the Corollary 3.10 and the Corollary 3.15 there is an equivalence of categories:

$$\mathfrak{Coh} X_{L, \vec{r}} \simeq \mathcal{EP}(X, L, \vec{r}),$$

so we reduced the problem to describing the  $K$ -theory of the (abelian) category of extendable pairs  $\mathcal{EP}(X, L, \vec{r})$ :

$$G(X_{L, \vec{r}}) \cong K(\mathcal{EP}(X, L, \vec{r})).$$

We are going to use localisations of the category of extendable pairs from the previous subsection to simplify the latter  $K$ -theory. The first step is the following

**Lemma 3.28.**

$$K_i(\mathcal{EP}(X, L, \vec{r})) \cong G_i(X) \oplus K_i(\mathbf{ker}(\pi_*^{L, \vec{r}})) \text{ for any } i \in \mathbb{Z}_+$$

*Proof.* Using Corollary 3.19 and the localisation property of  $K$ -theory (see for example [Q]) we have the long exact sequence of groups:

$$\dots \rightarrow K_i(\mathbf{ker}(\pi_*^{L, \vec{r}})) \rightarrow K_i(\mathcal{EP}(X, L, \vec{r})) \rightarrow G_i(X) \rightarrow \dots$$

But this sequence splits because of the property  $\pi_*^{L, \vec{r}} \circ \pi_{L, \vec{r}}^* \simeq 1$  proved in the Lemma 3.17.

□

Also we want to state the following

**Lemma 3.29.** *If  $\mathbf{A}$  is an abelian category then*

$$K_i(\mathrm{Func}(\vec{r}I^n, \mathbf{A})) \cong K_i(\mathbf{A})^{\oplus \prod_{j=1}^n r_j}.$$

*Proof.* The proof follows from the iterated application of the Theorem 2.6 and localisation property of the  $K$ -theory. □

Now we want to proceed with  $K_\bullet(\ker(\pi_*^{L,\vec{r}}))$  as in the previous lemmas. By combining the localisation property of the  $K$ -theory, Theorem 3.27 and Lemma 3.29 one can easily obtain the proof of the final

**Lemma 3.30.** *For any  $i \in \mathbb{Z}_+$*

$$K_i(\ker(\pi_*^{L,\vec{r}})) \cong \bigoplus_{k=1}^n \bigoplus_{T \in \mathcal{S}(k)} G_i\left(\bigcap_{l \in T} \operatorname{div}(s_l)\right)^{\oplus \prod_{l \in T} (r_l - 1)}.$$

We want to give sufficient conditions when a root stack is smooth.

**Proposition 3.31.** *Let  $X$  be a smooth scheme over a field  $k$ . Let  $D = \sum_{i=1}^n D_i$  be a normal crossing divisor. Assume that  $\vec{r}$  is an  $n$ -tuple of natural numbers, such that each  $r_i$  is coprime to the characteristic of  $k$ . Then a root stack  $X_{D,\vec{r}}$  is smooth.*

*Proof.* By definition a stack is smooth if its presentation is a smooth scheme. The question is local, so we can assume that  $X = \operatorname{Spec}(R)$  and a divisor  $D$  is a strict normal crossing divisor. If we localize further, we can assume that  $R$  is a local ring,  $D_i = (f_i)$  and  $f_i$  form a part of a regular sequence.

By Example [C, 2.4.1], the presentation of a root stack  $X_{D,\vec{r}}$  is an affine scheme  $A = R[t_1, \dots, t_n] / (t_1^{r_1} - f_1, \dots, t_n^{r_n} - f_n)$ . Under our assumptions this scheme is smooth. □

**Corollary 3.32.** *Under hypothesis of Proposition 3.31  $G(X_{D,\vec{r}}) = K(X_{D,\vec{r}})$ .*

*Proof.* Indeed, if a stack is regular, its  $K$ -theory is the same as  $G$ -theory. See [J]. □

## 4. QUOTIENT STACKS AS ROOT STACKS

**4.1. Generation of inertia groups.** Let  $X$  be a scheme with an action of a finite group  $G$ . We will always assume that this action is *admissible*, see [SGA1, V.1]. If  $x \in X$  is a point (not necessarily closed) the subgroup of  $G$  stabilising  $x$  is called the *decomposition group* and we denote it by  $D(x, G)$ . The subgroup of the decomposition group acting trivially on the residue field of  $x$  is called the *inertia group* of  $x$  and we denote it by  $I(x, G)$ .

Note that there is an induced action of  $D(x, G)$  on the closure of the point  $x$  and  $I(x, G)$  acts trivially on this closure. Hence if  $x \in \bar{y}$  then there is an inclusion  $I(y, G) \hookrightarrow I(x, G)$ . We will say that *the inertia groups are generated in codimension one* if for each point  $x \in X$  we



have that

$$I(x, G) = \prod_{x \in \bar{y}} I(y, G)$$

where the product is over all points of codimension one containing  $x$  and the identification is via the inclusions above. For a group acting on a smooth curve all inertia groups will be generated in codimension one. We are going to prove that under special assumption this will be also true in higher dimensions.

The important tool in studying quotients is Luna's étale slice theorem (see [D] and [L]). Roughly it says that étale-locally at a point  $x$  a  $G$ -action on  $X$  looks like a linear action of the decomposition group  $D(x, G)$  on a vector space  $N_x$ . Let us give the precise formulation for the reader's convenience.

**Theorem 4.1** (Luna's étale slice). *Let  $k$  be a field,  $X$  an affine regular variety, and  $G$  a finite group acting on  $X$ , such that its order is coprime to the characteristic of  $k$ . Consider a closed point  $x \in X$ . Then there exists a locally closed subscheme  $V$  of  $X$ , such that:*

- (1)  $V$  is affine and contains  $x$ .
- (2)  $V$  is  $G_x$ -invariant.
- (3) The image of  $G$ -morphism  $\psi : G \times_{G_x} V \rightarrow X$  is a saturated open subset  $U$  of  $X$ .
- (4) The restriction of  $\psi : G \times_{G_x} V \rightarrow U$  is strongly étale.

Also there exists an étale  $G_x$ -invariant morphism  $\phi : V \rightarrow T_x V$ , such that  $\phi(x) = 0$ ,  $T\phi_x = \text{Id}$ , and the following properties are satisfied:

- (5)  $T_x X = T_x(Gx) \oplus T_x V$ .
- (6) The image of  $\phi$  is a saturated open subset  $W$  of  $T_x V$ .
- (7) The restriction of  $\phi : V \rightarrow W$  is strongly étale  $G_x$ -morphism.

*Proof.* It is a slightly modified version of [D, Theorem 5.3] and [D, Theorem 5.4].  $\square$

Now we can give sufficient conditions for inertia groups to be generated in codimension one.

**Theorem 4.2.** *Let  $X$  be a regular, separated, noetherian scheme over a field  $k$ . Assume that  $G$  is a finite group with cardinality coprime to the characteristic of  $k$  and that  $G$  acts admissibly and generically freely on  $X$  with quotient  $\phi : X \rightarrow Y$  and  $Y$  is regular. Assume that the map  $\phi$  is ramified along a simple normal crossing divisor. For any point  $x \in X$  the inertia group  $I(x, G)$  is generated in codimension one.*

*Proof.* Before we start the proof let's observe that under assumption of the theorem the inertia group is preserved under arbitrary base change.

Consider a morphism  $f : Y' \rightarrow Y$  and  $X' = X \times_Y Y'$ , we want to show that  $I(x, G) = I(x', G)$  for a point  $x'$  over  $x$ . The inclusion  $I(x, G) \subset I(x', G)$  is clear. For the opposite inclusion notice that the quotient  $\phi : X \rightarrow Y$  is universal geometric quotient (see [FKM]), so we can assume that  $Y = \text{Spec}(k(y))$ , where  $y = \phi(x)$ . If we replace  $G$  by the decomposition group  $D(x, G)$ , we can also assume that  $X = \text{Spec}(k(x))$ . Now we can change  $G$  to  $G/I(x, G)$ . Then  $\phi : X \rightarrow Y$  will be a torsor, so is the base change.

This observation allows us to make a series of reductions. First using universality of the quotient we reduce the theorem to the case where  $X$  and  $Y$  are affine and the point  $x$  is closed. Also we can assume that the ground field  $k$  is separably closed, hence the inertia group  $I(x, G)$  equals to the decomposition group  $D(x, G)$ , which we will denote by  $G_x$ .

For the next reduction we are going to use Luna's étale slice theorem.

First we can replace  $X$  with an open subset  $U$  from part (3) of Theorem 4.1. Then we can make an étale base change and replace  $U$  with  $G \times_{G_x} V$  from part (4). We will get a cartesian diagram:

$$\begin{array}{ccc} G \times_{G_x} V & \longrightarrow & U \subset X \\ \downarrow & & \downarrow \\ V/G_x & \longrightarrow & U/G \end{array},$$

where the horizontal maps are étale. Hence the vertical map on the left has the same inertia as the map on the right.

Next, using part (7) of Theorem 4.1 we obtain a cartesian diagram:

$$\begin{array}{ccc} G \times_{G_x} V & \longrightarrow & G \times_{G_x} N_x \\ \downarrow & & \downarrow \\ V/G_x & \longrightarrow & N_x/G_x \end{array},$$

where the horizontal maps are étale and  $N_x$  is a vector space. The vertical map on the left has the same inertia as the map on the right.

We reduced the Theorem 4.2 to the statement that the decomposition subgroups of  $G$ -action on  $G \times_{G_x} N_x$  are generated in codimension one, where  $N_x$  is a vector space.

First notice that  $(G \times_{G_x} N_x)/G \cong N_x/G_x$ . Also  $N_x/G_x$  is regular as étale cover of a regular scheme. For any element  $u := \overline{(g, n)} \in G \times_{G_x} N_x$  the decomposition group  $G_u$  is equal to  $gH_n g^{-1}$  - the conjugate of the

decomposition group of  $n$  under  $G_x$ -action on  $N_x$  (by [D, Proposition 4.9]).

So finally, we have a linear action of  $G_x$  on a vector space  $N_x$ , the quotient  $N_x/G_x$  is regular and it ramifies along a simple normal crossing divisor. Chevalley-Sheppard-Todd theorem (see [Bou, Chapter V, §5, 5]) tells us that in this situation the group  $G_x$  is generated by pseudo-reflections.

Let's remind the reader that for a finite dimensional vector space  $V$  an element  $g \in GL(V)$  is called a *pseudo-reflection*, if the image  $\text{Im}(1 - g)$  is of dimension one. For a pseudo-reflection  $g$  we denote by  $V_g = \{v \in V \mid gv = v\}$ .

Finally we are left to prove a linear algebra fact:

**Lemma 4.3.** *Let  $g$  and  $h$  be pseudo-reflections and  $G = \langle g, h \rangle$  is a subgroup of  $GL(V)$  of finite order coprime to the characteristic of the ground field  $k$ . Denote by  $\Lambda$  the set of all pseudo-reflections in  $G$ . If  $\bigcup_{\tau \in \Lambda} V_\tau$  is supported on a strict normal crossing divisor, then the group  $G$  is abelian.*

*Proof.* It is enough to show that  $g$  and  $h$  commute. We have three pseudo-reflections  $g, h, hgh^{-1}$  and three hyperplanes  $V_g, V_h, V_{hgh^{-1}}$ . Firstly, observe that  $V_g \cap V_h = V_g \cap V_h \cap V_{hgh^{-1}}$ , because  $V_{hgh^{-1}} = hV_g$ .

The normal crossing condition forces  $V_{hgh^{-1}} = V_g$  or  $V_{hgh^{-1}} = V_h$ .

The condition  $V_{hgh^{-1}} = V_h$  is equivalent to  $hV_g = V_h$ . Let's apply  $h$  to both sides  $(\text{ord}(h) - 1)$  times, then we get  $V_h = V_g$ . Because  $V_g = V_h$  is fixed by  $G$ , we can find a  $G$ -invariant complement  $W$ . As  $\dim(W) = 1$  it is a subspace of common eigenvectors for  $G$ . This would imply that  $g$  and  $h$  commute.

Assume that  $V_{hgh^{-1}} = V_g$ , then  $hgh^{-1} = g$  (because they must commute and have the same characteristic polynomial). □

Hence we get that  $G_x$  is an abelian group. By definition, this means that the decomposition group of each point of  $N_x$  is generated in codimension one. □

**4.2. Main theorem.** In this subsection we will provide sufficient conditions for a quotient stack to be a root stack. To illustrate the procedure we will start with an example.

**Example 4.4.** Let  $\mathcal{O}$  be a discrete valuation ring with an action of  $\mu_r$  with  $\gcd(r, \text{char}(\mathcal{O})) = 1$ . Then the fixed ring  $\mathcal{O}^{\mu_r}$  is also a discrete valuation ring. We will assume that  $\mathcal{O}$  contains a field so that its completion  $\hat{\mathcal{O}}$  is a power series ring in one variable over the residue field.

Note that  $\mu_r$  must preserve the maximal ideal of  $\mathcal{O}$ . If we further assume that the action is generically free and inertial, i.e  $\mu_r$  acts trivially on the residue field then if  $s$  is a local parameter for  $\mathcal{O}$  we can conclude that  $t = s^r$  is a local parameter for  $R = \mathcal{O}^{\mu_r}$ .

We set  $Y = \operatorname{Spec}(R)$  and consider the root stack

$$\mathfrak{Y} = Y_{R,t,r} \rightarrow Y.$$

The parameter  $s$  induces a  $\mu_r$ -equivariant morphism

$$X \rightarrow \mathfrak{Y}$$

corresponding to the triple  $(\mathcal{O}, s, m)$  where  $m$  is the canonical isomorphism  $\mathcal{O}^r \rightarrow \mathcal{O}$ . We will show later (4.9) that this morphism is in fact étale. Using the 2 out of 3 property for étale maps we get that the natural morphism

$$X \times \mu_r \rightarrow X \times_{\mathfrak{Y}} X$$

is étale. To show that  $[X/\mu_r] \cong \mathfrak{Y}$  it suffices to show that this morphism is radicial (universally injective) and surjective. In other words we need to show that it is a bijection on  $K$ -points for each field  $K$ .

Given a pair of  $K$ -points  $a$  and  $b$  of  $X$  that give a  $K$ -point of  $X \times_Y X$  the fiber of

$$X \times_{\mathfrak{Y}} X \rightarrow X \times_Y X$$

over this point consists of the space of isomorphisms between  $a^*(\mathcal{O}, s, m)$  and  $b^*(\mathcal{O}, s, m)$  in  $\mathfrak{Y}$ . If the support of the  $K$ -points is the generic point of  $\mathcal{O}$  this is just a singleton and if the support is the closed point then the space is a bitorsor over  $\mu_r$ . At any rate the morphism above is seen to be an isomorphism. Hence in this case we have that

$$[X/\mu_r] \cong \mathfrak{Y}.$$

**Remark 4.5.** A  $\mu_r$ -bundle  $P$  on a scheme  $Y$  is equivalent to the data of an invertible sheaf  $\mathcal{K}$  and an isomorphism  $\phi : \mathcal{K}^r \rightarrow \mathcal{O}_Y$ . To construct  $P$  explicitly consider the sheaf of algebras  $\operatorname{Sym}^{\bullet} \mathcal{K}^{-1}$ . There is a distinguished global section  $T \in \mathcal{K}^{-r}$  given by  $(\phi \otimes 1_{\mathcal{K}^{-r}}(1))$ . Then

$$P = \operatorname{Spec} (\operatorname{Sym}^{\bullet} \mathcal{K}^{-1} / (T - 1)).$$

**Remark 4.6.** Suppose that there is on  $Y$  an invertible sheaf  $\mathcal{N}$  and an isomorphism  $\mathcal{N}^r \rightarrow \mathcal{L}$ . Then  $Y_{\mathcal{L},s,r}$  is a global quotient stack, see [C, 2.3.1 and 2.4.1] and [B, 3.4]. We will need this below, so let's recall some of the details. The coherent sheaf

$$\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{N}^{-1} \oplus \dots \oplus \mathcal{N}^{-(r-1)}$$

can be given the structure of an  $\mathcal{O}_Y$ -algebra via the composition

$$\mathcal{N}^{-r} \xrightarrow{\sim} \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_Y.$$

There is an action of  $\mu_r$  on this sheaf via the action of  $\mu_r$  on  $\mathcal{N}^{-1}$  given by scalar multiplication. Then  $Y_{\mathcal{L},s,r} = [\mathrm{Spec}(\mathcal{A})/\mu_r]$ . We will need the explicit morphism

$$Y_{\mathcal{L},s,r} \rightarrow [\mathrm{Spec}(\mathcal{A})/\mu_r]$$

below so let's describe it. Consider a morphism  $a : X \rightarrow Y$ . A morphism  $X \rightarrow Y_{\mathcal{L},s,r}$ , lifting  $a$ , is a triple  $(\mathcal{M}, t, \phi)$ . As per the previous remark the sheaf  $\mathcal{M}^{-1} \otimes \mathcal{N}$  gives a  $\mu_r$ -torsor. The torsor comes from the algebra

$$\mathcal{B} = \mathrm{Sym}^\bullet \mathcal{M} \otimes a^* \mathcal{N}^{-1} / (T - 1).$$

To produce an  $X$ -point of  $[\mathrm{Spec}(\mathcal{A})/\mu_r]$  we need to describe a  $\mu_r$ -equivariant map

$$a^* \mathcal{A} \rightarrow \mathcal{B}.$$

This map comes from the section  $t$  via :

$$t \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}) = \mathrm{Hom}(a^* \mathcal{N}^{-1}, \mathcal{M} \otimes a^* \mathcal{N}^{-1}).$$

This construction generalizes in the obvious way to a finite list of invertible sheaves with section.

**Assumption 4.7.** We will assume that  $X$  and  $Y$  are regular, separated, noetherian schemes over a field  $k$ . Let  $G$  be a finite group with cardinality coprime to the characteristic of  $k$ . We will assume that  $G$  acts admissibly and generically freely on  $X$  with quotient  $\phi : X \rightarrow Y$ . Note that by [GW, Theorem 14.126] our hypothesis imply that the quotient map  $X \rightarrow Y$  is flat.

Consider the map  $\phi : X \rightarrow Y$  which is faithfully flat and finite. Recall that the set of points of  $X$  where  $\phi$  is ramified is called the branch locus. It has a natural closed subscheme structure defined by  $\mathrm{supp}(\Omega_{X/Y})$ . Because the conditions of the purity theorem [AK, VI, Thm 6.8] are satisfied in our situation this closed subscheme will give rise to an effective Cartier divisor which is called the branch divisor. We can write this divisor as

$$R = \sum_{i=1}^n (r_i - 1) \left( \sum_{g \in G} g^* D_i \right),$$

where each  $D_i$  is a prime divisor. As  $G$  acts generically freely, passing to generic points of our regular variety produces a Galois extension with Galois group  $G$ . We can view the  $D_i$  as points of the scheme  $X$ . The multiplicities  $r_i$  are related to the inertia groups of  $D_i$  via

$$r_i = |I(D_i, G)|,$$

see [N, Ch. I.9].

We let  $E_i$  be the image of  $D_i$  under  $\phi$ . It is called the ramification divisor. We form the root stack

$$\mathfrak{Y} = Y_{((E_1, r_1), \dots, (E_n, r_n))}.$$

Note that we have assumed that the characteristic of our ground field is coprime to  $G$  and hence to each  $r_i$ . It follows, via a local calculation along the ring extension  $\mathcal{O}_{X, D_i} / \mathcal{O}_{Y, E_i}$  that we have  $\phi^*(E_i) = r_i(\sum_{g \in G} g^* D_i)$ . This allows us to lift  $\phi$  to produce a diagram

$$\begin{array}{ccc} X & & \\ \phi \downarrow & \searrow \psi & \\ Y & \xleftarrow{\pi} & \mathfrak{Y}. \end{array}$$

The morphism  $\psi$  is equivariant in the sense that precomposition with  $g \in G$  produces a two-commuting diagram. This gives us a morphism

$$[X/G] \rightarrow \mathfrak{Y}$$

that we would like to show is an isomorphism under our assumption (4.7).

In the proof below we will need to make use of

**Proposition 4.8.** (*Abhyankar's lemma*) *Let  $Y = \text{Spec}(A)$  be a regular local scheme and  $D = \sum_{1 \leq i \leq r} \text{div}(f_i)$  a divisor with normal crossings, so that the  $f_i$  form part of a regular system of parameters for  $Y$ . Set  $\bar{Y} = \text{Supp}(D)$  and let  $U = Y \setminus \bar{Y}$ . Consider  $V \rightarrow U$  an étale cover that is tamely ramified along  $D$ . If  $y_i$  are the generic points of  $\text{supp}(\text{div}(f_i))$  then  $\mathcal{O}_{Y, y_i}$  is a discrete valuation ring. If we let  $K_i$  be its field of fractions then as  $V$  ramifies tamely we have that*

$$V|_{K_i} = \text{Spec}\left(\prod_{j \in J_i} L_{ji}\right)$$

where the  $L_j$  are finite separable extensions of  $K_i$ . We let  $n_{ji}$  be the order of the inertia group of the Galois extension generated by  $L_{ji}$  and let

$$n_i = \text{lcm}_{j \in J_i} n_{ji},$$

and set

$$A' = A[T_1, \dots, T_r] / (T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r) \quad Y' = \text{Spec}(A').$$

Then the étale cover  $V' = V \times_X X'$  of  $U \times_X X'$  extends uniquely upto isomorphism to an étale cover of  $X'$ .

*Proof.* This is [SGA1, Expose XIII, proposition 5.2]. The proof given shows how to construct the extension of  $V'$ , we will need this below. The extension can be constructed as the normalization of  $X'$  in the generic point of  $V \times_X X'$ .  $\square$

**Proposition 4.9.** *Suppose that  $X \rightarrow Y$  is ramified along a simple normal crossings divisor. The morphism  $\psi : X \rightarrow \mathfrak{Y}$  constructed above is étale.*

*Proof.* Étale maps are local on the source so we can assume that  $Y = \text{Spec}(S)$ , and all  $E_i$  are trivial line bundles so that  $s_i \in S$ . Further, by shrinking  $X$  we can assume that the morphism  $X \rightarrow \mathfrak{Y}$  is defined by trivial bundles on  $X$ . Because the map  $\phi$  is finite we can write  $X = \text{Spec}(T)$ . Here  $T$  and  $S$  are local regular Noetherian  $k$ -algebras,  $T$  is a finite  $S$ -module,  $s_i$  is part of a regular system of parameters and there are elements  $t_i \in T$ , such that  $t_i^{r_i} = s_i$ .

We may check étaleness after a faithfully flat base extension of the base field and hence may assume that the ground field  $k$  contains  $r_i$ -th roots of unity for all  $1 \leq i \leq n$ .

Using (4.6) the stack  $\mathfrak{Y}$  is isomorphic to the quotient stack

$$[\text{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}],$$

where  $S' = S[y_1, \dots, y_n]/(y_1^{r_1} - s_1, \dots, y_n^{r_n} - s_n)$ .

We want to show that the map  $\text{Spec}(T) \rightarrow [\text{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}]$  is étale. Denote by  $T'$  the ring  $T[x_1, \dots, x_n]/(x_1^{r_1} - 1, \dots, x_n^{r_n} - 1)$ . Using (4.6) again we have see that we have a Cartesian diagram :

$$\begin{array}{ccc} \text{Spec}(T') & \xrightarrow{\quad\quad\quad} & \text{Spec}(S') \\ \downarrow & & \downarrow \\ \text{Spec}(T) & \xrightarrow{\quad\quad\quad} & [\text{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}] \end{array}$$

Because  $\text{Spec}(S')$  is a presentation of a quotient stack it is enough to show that the map  $S' \rightarrow T'$  given by  $y_i \mapsto t_i x_i$  is étale.

The morphism  $S_{s_1 \dots s_n} \rightarrow T_{t_1 \dots t_n}$  is flat and unramified by assumption, hence it is étale. By Abhyankar's lemma, (4.8), this morphism extends after base change to an étale cover of  $S'$ . By the proof of Abhyankar's lemma it suffices to show that  $T'$  is normal and the map  $S' \rightarrow T'$  is integral. Both of these facts are easily checked and the result follows.  $\square$

For a point  $p \in Y$  we define

$$I(p, Y) = \prod_{p \in \text{supp}(E_i)} \mu_{r_i}.$$

**Proposition 4.10.** *Let  $K$  be a field and consider the morphism of  $K$ -points*

$$\pi_K : X \times_{\mathfrak{Y}} X(K) \rightarrow X \times_Y X(K).$$

*The fiber  $\pi_K^{-1}(x_1, x_2)$  over a  $K$ -point  $(x_1, x_2)$  is a bi-torsor under the inertia group  $I(\phi(x_1), Y)$ .*

*Proof.* In what follows, we will use the shorthand  $G^*$  when we mean  $\sum_{g \in G} g^*$ . Recall that the morphism  $\psi$  is defined by  $(\mathcal{O}(G^*E_i), s_{G^*E_i}, \alpha_i)$  where  $\alpha_i$  are isomorphisms coming from the fact that

$$r_i G^*E_i = r_i \phi^*(D_i).$$

The fiber over  $(x_1, x_2)$  is exactly the set of isomorphism from  $x_1^* \mathcal{O}(G^*E_i)$  to  $x_2^* \mathcal{O}(G^*E_i)$  as  $i$  varies. As in (4.4) this depends on whether the section  $x_1^* s_{G^*E_i}$  vanishes or not. The vanishing condition precisely depends on  $\phi(x_1)$  and the result follows.  $\square$

**Theorem 4.11.** *If the assumption 4.7 satisfies and if additionally the ramification divisor  $D$  is a normal crossing divisor then we have the isomorphism of stacks  $[X/G] \cong \mathfrak{Y}$ .*

*Proof.* To prove this all we need to show is that the map

$$\begin{aligned} \chi : X \times G &\rightarrow X \times_{\mathfrak{Y}} X \\ (x, g) &\mapsto (x, gx) \end{aligned}$$

is an isomorphism.

Using (4.9), the map  $\psi : X \rightarrow \mathfrak{Y}$  is étale, and so the map  $X \times_{\mathfrak{Y}} X \rightarrow X$  is étale as a pullback. Clearly two maps  $X \times G \rightarrow X$  given by  $(x, g) \mapsto x$  and  $(x, g) \mapsto gx$  are étale and so the map  $\chi$  must be étale.

We are going to show that the map

$$\chi(K) : X(K) \times G \rightarrow X \times_{\mathfrak{Y}} X(K)$$

is bijective for any field extension of the ground field  $k \subset K$ . The points of the scheme on the left is a pair  $(x, g)$ , where  $g \in G$  and  $x : \text{Spec}(K) \rightarrow X$  a  $K$ -point.

Consider the morphism  $\Psi : X \times G \rightarrow X \times_Y X$ . This morphism is surjective as we have a geometric quotient, see [FKM, Definition 0.4]. Consider a  $K$ -point  $(x_1, x_2) \in X \times_Y X(K)$ . Using the properties of geometric quotients we have that  $x_2 = gx_1$  for some  $g \in G$ . Using this we see the fiber  $\Psi^{-1}(x_1, x_2)$  is a torsor over the inertia group



$I(\text{supp}(x_1), G)$ . By Theorem 4.2 our inertia groups are generated in codimension one, so we see that we have an identification

$$I(\text{supp}(x_1), G) = \mu_{r_{i_1}} \times \dots \times \mu_{r_{i_l}}$$

as in the previous proposition. It follows that the morphism  $\chi$  is étale and universally injective (radicial). This implies that it is an open immersion. As it is also surjective it is an isomorphism and the result follows.  $\square$

## 5. APPLICATION OF ROOT STACKS TO EQUIVARIANT $K$ -THEORY OF SCHEMES

As an application of the theorems proved in the sections 3 and 4 we can formulate a result about equivariant  $K$ -theory.

**Theorem 5.1.** *Let  $X$  be a regular, separated, noetherian scheme over the field  $k$  with a generically free admissible action of a finite group  $G$ , such that the order of  $G$  is coprime to the characteristic of  $k$ . Let's denote  $X/G = Y$  and assume all the condition from the Assumption 4.7. Also assume that  $X \rightarrow Y$  is ramified along a simple normal crossing divisor. Then there is an isomorphism of groups:*

$$K_G^\bullet(X) \cong K^\bullet(Y) \oplus (\oplus_{i=1}^n (\oplus_{T \in S(i)} G^\bullet(\bigcap_{l \in T} E_l)^{\oplus \prod_{l \in T} (r_l - 1)})),$$

where  $E$  is a ramification divisor and  $r_l$  are orders of inertia groups (see Section 4 for notation).

*Proof.* By the assumptions  $X$  is a regular scheme and the group  $G$  is finite so for any  $G$ -equivariant sheaf we can always construct an equivariant locally free resolution by averaging the usual locally free resolution. This simple argument shows that equivariant  $K$ -theory of  $X$  should be the same as equivariant  $G$ -theory.

The category of  $G$ -equivariant sheaves on  $X$  is equivalent to the category of sheaves on the quotient stack  $[X/G]$  so we can see that

$$K_G(X) \cong G([X/G]).$$

In the Theorem 4.11 we proved that under our assumptions there is an isomorphism of stacks  $[X/G] \cong \mathfrak{Y}$ , so we have an isomorphism of their  $G$ -theories:

$$G([X/G]) \cong G(\mathfrak{Y}).$$

Finally the application of Lemma 3.28 and Lemma 3.30 gives formula we wanted to proof.  $\square$

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